Properties of Riemann Integral and Class of Integrable Functions

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Let $[a, b] \subseteq \mathbb{R}$ be a closed and bounded interval, $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, and $f : [a, b] \to \mathbb{R}$ be bounded function. We use the familiar notation
$M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}$, and $m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}$.
Note that

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k) \triangle x_k$$

Theorem 2.1: ***[Integrability of Monotone Functions]***

Let $f : [a, b] \to \mathbb{R}$ be monotone function on $[a, b]$. Then f is integrable on $[a, b]$.

Discussion: Suppose f is increasing on $[a, b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ such that $x_k - x_{k-1} = \dfrac{b-a}{n}$ for $k = 1, 2, \ldots, n$. Since f is increasing on $[x_{k-1}, x_k]$, then $m_k = f(x_{k-1})$ and $M_k = f(x_k)$. Hence

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M_k - m_k) \triangle x_k \\
&= \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] \left[\frac{b-a}{n}\right] \\
&= \frac{b-a}{n} \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] \\
&= \frac{b-a}{n} \left[(\cancel{f(x_1)} - f(x_0)) + (\cancel{f(x_2)} - \cancel{f(x_1)}) + \ldots + (f(x_n) - \cancel{f(x_{n-1})})\right] \\
&= \frac{b-a}{n} [f(x_n) - f(x_0)] \\
&= \frac{b-a}{n} [f(b) - f(a)] .
\end{aligned}$$

So we need to choose $n \in \mathbb{N}$ such that $\dfrac{b-a}{n}[f(b) - f(a)] < \varepsilon$.

Proof: Suppose f is increasing on $[a, b]$. Choose $n \in \mathbb{N}$ such that $\dfrac{b-a}{n}[f(b) - f(a)] < \varepsilon$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ such that $x_k - x_{k-1} = \dfrac{b-a}{n}$ for $k = 1, 2, \ldots, n$. Now,

$$U(f, P) - L(f, P) = \frac{b-a}{n} [f(b) - f(a)] < \varepsilon.$$

Hence f is integrable.

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Theorem 2.2: ***[Integrability of Continuous Functions]***

Let $f : [a, b] \to \mathbb{R}$ be continuous function on $[a, b]$. Then f is integrable on $[a, b]$.

Proof: Since f is continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$. Hence given $\varepsilon > 0$ there exist $\delta > 0$ such that if $x, y \in [a,b]$ and $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \dfrac{\varepsilon}{b-a}$. Now, choose $n \in \mathbb{N}$ such that $\dfrac{b-a}{n} < \delta$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ such that $x_k - x_{k-1} = \dfrac{b-a}{n}$ for $k = 1, 2, \ldots, n$. Since f is continuous on $[x_{k-1}, x_k]$, then f takes its maximum and minimum at some points in $[x_{k-1}, x_k]$. Hence $m_k = f(u_k)$ and $M_k = f(v_k)$ for some $u_k, v_k \in [x_{k-1}, x_k]$.
Hence $|v_k - u_k| \leq |x_k - x_{k-1}| = \dfrac{b-a}{n} < \delta \Rightarrow |f(v_k) - f(u_k)| < \dfrac{\varepsilon}{b-a}$.
Therefore

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n} (M_k - m_k) \triangle x_k \\ &= \sum_{k=1}^{n} [f(v_k) - f(u_k)] \left[\frac{b-a}{n}\right] \\ &< \frac{b-a}{n} \sum_{k=1}^{n} \frac{\varepsilon}{b-a} \\ &= \frac{b-a}{n} \frac{n\varepsilon}{b-a} \\ &= \varepsilon. \end{aligned}$$

Hence f is integrable.

Theorem 2.3: *[linearity I]*

Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. If $c \in \mathbb{R}$, then cf is integrable on $[a,b]$, and

$$\int_a^b cf = c \int_a^b f.$$

Proof: **Case 1:** $c > 0$.
Let $I \subseteq [a,b]$. Now, since

$$\inf\{(cf)(x) : x \in I\} = c \inf\{f(x) : x \in I\} \text{ and } \sup\{(cf)(x) : x \in I\} = c \sup\{f(x) : x \in I\},$$

then

$$U(cf,P) = cU(f,P) \text{ and } L(cf,P) = cL(f,P) \quad \text{for any partition } P \text{ of } [a,b].$$

Let $\varepsilon > 0$ be given. Since f is integrable $\exists$ a partition P_ε such that $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \dfrac{\varepsilon}{c}$.
Now,

$$\begin{aligned} U(cf,P_\varepsilon) - L(cf,P_\varepsilon) &= cU(f,P_\varepsilon) - cL(f,P_\varepsilon) \\ &= c[U(f,P_\varepsilon) - L(f,P_\varepsilon)] \\ &< c\frac{\varepsilon}{c} \\ &= \varepsilon. \end{aligned}$$

Hence cf is integrable. Now,

$$cL(f,P_\varepsilon) = L(cf,P_\varepsilon) \le \int_a^b cf \le U(cf,P_\varepsilon) = cU(f,P_\varepsilon) \qquad (1)$$

and

$$L(f,P_\varepsilon) \le \int_a^b f \le U(f,P_\varepsilon).$$

If we multiply the last inequality by $c > 0$ we get

$$cL(f,P_\varepsilon) \le c\int_a^b f \le cU(f,P_\varepsilon).$$

Now, using $U(cf,P) = cU(f,P)$ and $L(cf,P) = cL(f,P)$ in the last inequality we have

$$L(cf,P_\varepsilon) \le c\int_a^b f \le U(cf,P_\varepsilon).$$

Hence

$$-U(cf,P_\varepsilon) \le -c\int_a^b f \le -L(cf,P_\varepsilon) \qquad (2)$$

Adding (1) and (2) we get

$$-[U(cf,P_\varepsilon) - L(cf,P_\varepsilon)] \le \int_a^b cf - c\int_a^b f \le U(cf,P_\varepsilon) - L(cf,P_\varepsilon) \Rightarrow \left|\int_a^b cf - c\int_a^b f\right| < U(cf,P_\varepsilon) - L(cf,P_\varepsilon) < \varepsilon.$$

Hence

$$\left|\int_a^b cf - c\int_a^b f\right| = 0 \Rightarrow \int_a^b cf - c\int_a^b f = 0 \Rightarrow \int_a^b cf = c\int_a^b f$$

Case 2: $c = 0$.

The function $cf = 0$ which is integrable and

$$\int_a^b cf = \int_a^b 0 = 0 = 0.\int_a^b f = c\int_a^b f.$$

Case 3: $c < 0$.

Try to do this case.

Theorem 2.4: *[linearity II]*

Let $f, g : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then $f+g$ is integrable on $[a,b]$, and

$$\int_a^b f+g = \int_a^b f + \int_a^b g.$$

Proof:

Let $I \subseteq [a,b]$. Now, since

$$\inf\{(f+g)(x) : x \in I\} \ge \inf\{f(x) : x \in I\} + \inf\{g(x) : x \in I\}$$

and

$$\sup\{(f+g)(x) : x \in I\} \leq \sup\{f(x) : x \in I\} + \sup\{g(x) : x \in I\},$$

then

$$U(f+g,P) \leq U(f,P) + U(g,P) \text{ and } L(f,P) + L(g,P) \leq L(f+g,P) \text{ for any partition } P \text{ of } [a,b].$$

Let $\varepsilon > 0$ be given. Since f and g are integrable $\exists$ a partitions P_1 and P_2 such that $U(f,P_1) - L(f,P_1) < \dfrac{\varepsilon}{2}$ and $U(f,P_2) - L(f,P_2) < \dfrac{\varepsilon}{2}$.
Now, let

$$P_\varepsilon = P_1 \cup P_2 \Rightarrow U(f+g,P_\varepsilon) \leq U(f,P_\varepsilon) + U(g,P_\varepsilon) \text{ and } L(f,P_\varepsilon) + L(g,P_\varepsilon) \leq L(f+g,P_\varepsilon)$$

also,

$$P_\varepsilon = P_1 \cup P_2 \Rightarrow U(f,P_\varepsilon) \leq U(f,P_1), \quad L(f,P_1) \leq L(f,P_\varepsilon), \quad U(g,P_\varepsilon) \leq U(g,P_2), \quad \text{and, } L(g,P_2) \leq L(g,P_\varepsilon).$$

Now,

$$\begin{aligned} U(f+g,P_\varepsilon) - L(f+g,P_\varepsilon) &\leq U(f,P_\varepsilon) + U(g,P_\varepsilon) - [L(f,P_\varepsilon) + L(g,P_\varepsilon)] \\ &\leq [U(f,P_\varepsilon) - L(f,P_\varepsilon)] + [U(g,P_\varepsilon) - L(g,P_\varepsilon)] \\ &\leq [U(f,P_1) - L(f,P_1)] + [U(g,P_2) - L(g,P_2)] \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $f+g$ is integrable. Now,

$$\int_a^b f+g \leq U(f+g,P_\varepsilon) \leq U(f,P_\varepsilon) + U(g,P_\varepsilon) \leq L(f,P_\varepsilon) + L(g,P_\varepsilon) + \varepsilon \leq \int_a^b f + \int_a^b g + \varepsilon.$$

Hence

$$\int_a^b f+g - (\int_a^b f + \int_a^b g) \leq \varepsilon \qquad (1).$$

Also,

$$\int_a^b f + \int_a^b g \leq U(f,P_\varepsilon) + U(g,P_\varepsilon) \leq L(f+g,P_\varepsilon) + \varepsilon \leq \int_a^b f+g+\varepsilon.$$

Hence

$$-\varepsilon \leq \int_a^b f+g - (\int_a^b f + \int_a^b g) \qquad (2).$$

By (1) and (2) we have

$$-\varepsilon \leq \int_a^b f+g - (\int_a^b f + \int_a^b g) \leq \varepsilon.$$

Hence

$$\left|\int_a^b f+g - (\int_a^b f + \int_a^b g)\right| \leq \varepsilon \Rightarrow \left|\int_a^b f+g - (\int_a^b f + \int_a^b g)\right| = 0 \Rightarrow \int_a^b f+g = \int_a^b f + \int_a^b g.$$

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Theorem 2.5: *[]*

If f and g are integrable on $[a,b]$ and if $f(x) \leq g(x)$ for $x \in [a,b]$, then $\int_a^b f \leq \int_a^b g$.

Proof: Let $h(x) = g(x) - f(x)$ for all $x \in [a,b]$. Then by the linearity of Riemann integrable h is integrable. Now, $h(x) \geq 0$ for all $x \in [a,b]$, hence $0 \leq m = \inf\{h(x) : x \in [a,b]\} \leq m_i = \inf\{h(x) : x \in [x_{i-1}, x_i] \subset [a,b]\}$. Thus if P is any partition of $[a,b]$, we have $0 \leq L(h,P) \leq \int_a^b h$.

$$\begin{aligned} 0 &\leq L(h,P) \\ &\leq \int_a^b h \qquad h(x) = g(x) - f(x) \\ &= \int_a^b (g-f) \quad \text{by the linearity of the Riemann integral} \\ &= \int_a^b g - \int_a^b f \end{aligned}$$

$$\int_a^b f \leq \int_a^b g$$

Theorem 2.6: ***[Integrability of the absolute value of an integrable function]***

Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$, and

$$|\int_a^b f| \leq \int_a^b |f|.$$

Proof:

Let $I \subseteq [a,b]$. Now, since We have

$$\sup\{|f(x)| : x \in I\} - \inf\{|f(x)| : x \in I\} \leq \sup\{f(x) : x \in I\} - \inf\{f(x) : x \in I\}.$$

Let $\varepsilon > 0$ be given. Since f is integrable $\exists$ a partition P_ε such that $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$

Now, $U(|f|,P_\varepsilon) - L(|f|,P_\varepsilon) \leq U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$. Hence $|f|$ is integrable. Now,

$$f(x) \leq |f(x)| \Rightarrow \int_a^b f(x) \leq \int_a^b |f(x)| \qquad -f(x) \leq |f(x)| \Rightarrow -\int_a^b f(x) \leq \int_a^b |f(x)|.$$

Hence $-\int_a^b |f(x)| \leq \int_a^b f(x) \leq \int_a^b |f(x)|$ Therefore $\left|\int_a^b f(x)\right| \leq \int_a^b |f(x)|$.

Theorem 2.7: ***[Integrability of the squarer of an integrable function]***

Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then f^2 is integrable on $[a,b]$.

Proof:

Since f is bounded ,then there exist $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.

Now,let $I \subseteq [a,b]$. We have

$$\sup\{f^2(x) : x \in I\} - \inf\{f^2(x) : x \in I\} \leq 2M[\sup\{|f(x)| : x \in I\} - \inf\{|f(x)| : x \in I\}]$$

Let $\varepsilon > 0$ be given. Since $|f|$ is integrable $\exists$ a partition P_ε such that $U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) < \dfrac{\varepsilon}{2M}$

Now,

$$U(f^2, P_\varepsilon) - L(f^2, P_\varepsilon) \leq 2MU(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) < 2M\frac{\varepsilon}{2M} = \varepsilon.$$

Hence f^2 is integrable.

