EUCLIDEAN SPACES

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Definition 0.1. for each $n \in \mathbb{N}$ we define the Euclidean space, $\mathbb{R}^n$, by

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n\}.$$

Elements $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of $\mathbb{R}^n$ is called *vectors (points)* and each number x_i is called the *ith coordinate* or *components* of $\mathbf{x}$.

Definition 0.2. Algebraic Structure of $\mathbb{R}^n$ Let $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ be vectors and $\alpha \in \mathbb{R}$ be a scalar.

(1) $\mathbf{x} = \mathbf{y}$ if and only if $x_i = y_i$ for $i = 1, 2, \ldots, n$. (Two vectors are equal if their components are equal)

(2) The zero vector is $\mathbf{0} = (0, 0, \ldots, 0)$.

(3) The sum of $\mathbf{x}$, and $\mathbf{y}$ is the vector $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

(4) The difference of $\mathbf{x}$, and $\mathbf{y}$ is the vector $\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$.

(5) The product of a scalar α and a vector $\mathbf{x}$ is the vector $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$.

(6) The dot product of $\mathbf{x}$ and $\mathbf{y}$ is the scalar $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n$.

(7) The vector $\mathbf{e}_i = (0, 0, 0, \ldots, 1^{ith \text{ component}}, 0, 0, \ldots, 0)$

(8) The usual basis of $\mathbb{R}^n$ is the collection $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

(9) The Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is the scalar $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

(10) The Euclidean distance between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is the scalar $\|\mathbf{x} - \mathbf{y}\|$.

(11) The sup-norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is the scalar $\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$.

Theorem 0.1. *The Cauchy-Schwarz Inequality* *If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then $|\boldsymbol{x} \cdot \boldsymbol{y}| \le \|\boldsymbol{x}\|\|\boldsymbol{y}\|$.*

Proof. If $\mathbf{y} = 0$ there is nothing to prove. Suppose $\mathbf{y} \neq 0$. Now, $\alpha = \dfrac{(\mathbf{x} \cdot \mathbf{y})}{\|\mathbf{y}\|^2} \in \mathbb{R}$. Now,

$$0 \le \|\mathbf{x} - \alpha\mathbf{y}\|^2 = (\mathbf{x} - \alpha\mathbf{y}) \cdot (\mathbf{x} - \alpha\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} - \alpha\mathbf{x} \cdot \mathbf{y} - \alpha\mathbf{x} \cdot \mathbf{y} + \alpha\mathbf{y} \cdot \alpha\mathbf{y} = \|\mathbf{x}\|^2 - 2\alpha\mathbf{x} \cdot \mathbf{y} + \alpha^2\|\mathbf{y}\|^2$$

Hence $0 \le \|\mathbf{x}\|^2 - 2\alpha(\mathbf{x} \cdot \mathbf{y}) + \alpha^2\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\dfrac{(\mathbf{x} \cdot \mathbf{y})}{\|\mathbf{y}\|^2}(\mathbf{x} \cdot \mathbf{y}) + \left(\dfrac{(\mathbf{x} \cdot \mathbf{y})}{\|\mathbf{y}\|^2}\right)^2 \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - \dfrac{(\mathbf{x} \cdot \mathbf{y})^2}{\|\mathbf{y}\|^2}$

Thus $\dfrac{(\mathbf{x} \cdot \mathbf{y})^2}{\|\mathbf{y}\|^2} \le \|\mathbf{x}\|^2$. Hence $(\mathbf{x} \cdot \mathbf{y})^2 \le \|\mathbf{x}\|^2\|\mathbf{y}\|^2 = (\|\mathbf{x}\|\|\mathbf{y}\|)^2$. By taking the square root of both sides we get $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\|\|\mathbf{y}\|$.

□

Theorem 0.2. *Let* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. *Then*

(i) $\|\boldsymbol{x}\| \geq 0$ *with equality only when* $\boldsymbol{x} = \boldsymbol{0}$,

(ii) $\|\alpha \boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\|$ *for all scalars* α,

(iii) $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$, *(Triangle Inequality)*

(iv) $|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x}\| - \|\boldsymbol{y}\|$,

(v) $\|\boldsymbol{x}\| \leq \sum_{k=1}^{n} |x_k| \leq n\|\boldsymbol{x}\|_\infty$,

(vi) $|x_k| \leq \|\boldsymbol{x}\| \leq \sqrt{n}\|\boldsymbol{x}\|_\infty$.

Proof. We will prove $iii - vi$, you should be able to do $i - ii$.

$$
\begin{aligned}
(iii) \quad \|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) &= \mathbf{x}\cdot\mathbf{x} + 2(\mathbf{x}\cdot\mathbf{y}) + \mathbf{y}\cdot\mathbf{y} \\
&= \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2 \\
&\leq \|\mathbf{x}\|^2 + 2|\mathbf{x}\cdot\mathbf{y}| + \|\mathbf{y}\|^2 \\
&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \quad \text{by the Cauchy-Schwarz Inequality} \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \\
\text{Hence } \|\mathbf{x}+\mathbf{y}\|^2 &\leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2
\end{aligned}
$$

By taking the square root, we get $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

$$
\begin{aligned}
(iv) \qquad \|\mathbf{x}\| = \|\mathbf{x}-\mathbf{y}+\mathbf{y}\| &\leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}\| \quad \text{by Triangle Inequality } (iii) \\
\text{Hence } \|\mathbf{x}\| - \|\mathbf{y}\| &\leq \|\mathbf{x}-\mathbf{y}\| -----------------(1) \\
\text{Also } \|\mathbf{y}\| = \|\mathbf{y}-\mathbf{x}+\mathbf{x}\| &\leq \|\mathbf{y}-\mathbf{x}\| + \|\mathbf{x}\| \quad \text{by Triangle Inequality } (iii) \\
\text{Hence } \|\mathbf{y}\| - \|\mathbf{x}\| &\leq \|\mathbf{y}-\mathbf{x}\| = \|-(\mathbf{x}-\mathbf{y})\| = \|\mathbf{x}-\mathbf{y}\| \\
\text{Hence } \; -\|\mathbf{x}-\mathbf{y}\| &\leq \|\mathbf{x}\| - \|\mathbf{y}\| -----------------(2)
\end{aligned}
$$

By (1) and (2) we get $-\|\mathbf{x}-\mathbf{y}\| \leq \|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\| \Rightarrow |\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x}\| - \|\mathbf{y}\|$.

$$
(v) \quad \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \ldots + x_n^2 = |x_1|^2 + |x_2|^2 + \ldots + |x_n|^2 \leq (|x_1| + |x_2| + \ldots + |x_n|)^2
$$

Hence $\|\mathbf{x}\| \leq (|x_1| + |x_2| + \ldots + |x_n|) = \displaystyle\sum_{k=1}^{n} |x_k|$

Now, since $|x_k| \leq \displaystyle\max_{1\leq k\leq n} |x_k| = \|\mathbf{x}\|_\infty$ for $k = 1, 2, \ldots, n$ then $\|\mathbf{x}\| \leq \displaystyle\sum_{k=1}^{n} |x_k| \leq \sum_{k=1}^{n} \|\mathbf{x}\|_\infty = n\|\mathbf{x}\|_\infty$.

$$
(vi) \quad |x_k|^2 \leq \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \ldots + x_n^2 = |x_1|^2 + |x_2|^2 + \ldots + |x_n|^2 = \sum_{k=1}^{n} |x_k|^2 \quad \text{for } k = 1, 2, \ldots, n
$$

Hence $|x_k|^2 \leq \|\mathbf{x}\|^2 \leq \displaystyle\sum_{k=1}^{n} |x_k|^2 \leq \sum_{k=1}^{n} \|\mathbf{x}\|_\infty^2 = n\|\mathbf{x}\|_\infty^2$

Thus $|x_k| \leq \|\mathbf{x}\| \leq \sqrt{n}\,\|\mathbf{x}\|_\infty$.

□

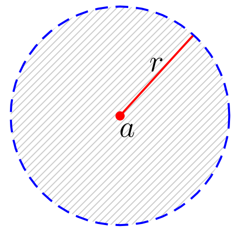

FIGURE 1

Definition 0.3. Let $a \in \mathbb{R}^n$ and $r > 0$. We define the *open ball* of $\mathbb{R}^n$ to be the set

$$\boldsymbol{B_r(a) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - a\| < r\}.}$$

Definition 0.4. Let $E \subseteq \mathbb{R}^n$ We say that E is open set if for each $\mathbf{x} \in E$ there is an $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subseteq E$.

Lemma 0.1. *Every open ball in $\mathbb{R}^n$ is open.*

Proof. Let $\mathbf{x} \in B_r(a)$ and let $\epsilon = r - \|\mathbf{x} - a\|$. We claim that $B_\epsilon(\mathbf{x}) \subseteq B_r(a)$. So let $\mathbf{y} \in B_\epsilon(\mathbf{x})$, then $\|\mathbf{y} - \mathbf{x}\| < \epsilon$. Now, $\boldsymbol{\|\mathbf{y} - a\| = \|\mathbf{y} - \mathbf{x} + \mathbf{x} - a\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - a\| < \epsilon + \|\mathbf{x} - a\| = r - \|\mathbf{x} - a\| + \|\mathbf{x} - a\|\| = r.}$ Thus $\|\mathbf{y} - a\| < r$. Hence $\mathbf{y} \in B_r(a)$. Therefore $B_\epsilon(\mathbf{x}) \subseteq B_r(a)$ and hence $B_r(a)$ is open set. $\square$

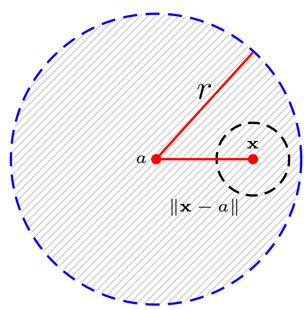

FIGURE 2

Example 0.1. The set $E = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < 1\}$ is open set since for every $\mathbf{x} \in E$ we can find an open ball contained in E.

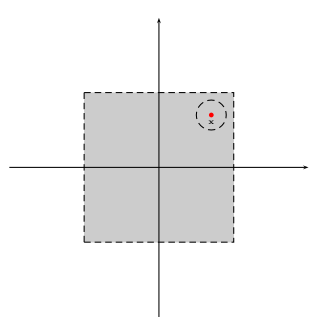

FIGURE 3

Definition 0.5. Let $E \subseteq \mathbb{R}^n$. We say that E is closed set if $E^c = \mathbb{R}^n \setminus E$ is open set.

Lemma 0.2. *Every singleton in $\mathbb{R}^n$ is closed. (Let $\boldsymbol{x} \in \mathbb{R}^n$, then $\{\boldsymbol{x}\}$ is closed set.)*

Proof. We want to show that $\{\mathbf{x}\}^c = \mathbb{R}^n \setminus \{\mathbf{x}\}$ is open. Let $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{x}\}$, then $\|\mathbf{x} - \mathbf{y}\| > 0$. Let $r = \dfrac{\|\mathbf{x} - \mathbf{y}\|}{2}$, then $B_r(\mathbf{y}) \cap \{\mathbf{x}\} = \phi \Rightarrow B_r(\mathbf{y}) \subseteq \mathbb{R}^n \setminus \{\mathbf{x}\}$. Hence $\{\mathbf{x}\}^c = \mathbb{R}^n \setminus \{\mathbf{x}\}$ is open set. Therefore $\{\mathbf{x}\}$ is closed. □

Definition 0.6. Let $E \subseteq \mathbb{R}^n$, and let $\mathbf{x} \in \mathbb{R}^n$.

- We say that $\mathbf{x}$ is an interior point of E if there exist $r > 0$ such that $B_r(\mathbf{x}) \subseteq E$.
- The set of all interior points of E is denoted by E°.
- We say that $\mathbf{x}$ is a limit point of E if for each $r > 0$, $B_r(\mathbf{x}) \cap (E \setminus \{\mathbf{x}\}) \neq \phi$.
- The set of all limit points of E is denoted by E'.
- We say that $\mathbf{x}$ is a boundary point of E if for each $r > 0$, $B_r(\mathbf{x}) \cap E \neq \phi$ and $B_r(\mathbf{x}) \cap E^c \neq \phi$.
- The set of all boundary points of E is denoted by ∂E.
- The closure set of E ,denoted by $\overline{E}$, is $\overline{E} = E \cup E'$.

Example 0.2. Let $E = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < 1\}$. Then every point of E is an interior point and $E^\circ = E$. and every point in E and on the boundary of E is a limit point and $E' = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Hence $\overline{E} = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Every point on the lines $x = 1, x = -1, y = 1, y = -1, - \leq x, y \leq 1$. is a boundary point and
$\partial E == \{(x, y) \in \mathbb{R}^2 \mid y = \pm 1, -1 \leq x \leq 1 \text{ and } x = \pm 1, -1 \leq y \leq 1\}\ X \times Y$